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CANONICAL TENSOR IN THE THEORY OF ELASTICITY

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The study [1] presented a noncanonical form of a symmetrical tensor whereby the tensor assumes the simplest possible (diagonal) form in principal axes. Here, we define a symmetrical tensor which changes the quadratic form of the potential energy in a unit volume of an elastic body to canonical form. It is shown that such a transformation can be made by an appropriate selection of two constants in a form analogous to the generalized Hooke's law.

The stress–strain state in an elementary volume of an elastic body is characterized by the stress tensor σ_{ij} and the strain tensor ε_{ij} ($i, j = 1, \ldots, 3$). The components of these tensors are connected by the elasticity relations

$$\sigma_{ij} = b_{ijkm}\varepsilon_{km}. \tag{1}$$

Here and below, b_{ijkm} is the tensor of the elastic constants. Summation is carried out over twice-repeated subscripts. In an isotropic elastic body

$$b_{ijkm} = \lambda\delta_{ij}\delta_{km} + \mu(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}), \tag{2}$$

where λ and μ are the Lamé constants; δ_{ij} is the Kronecker symbol.

We will define the canonical tensor s_{ij} as a tensor having components connected with the components of the strain tensor by the same relations that connect the components of the stress tensor with the components of the canonical tensor, i.e.,

$$s_{ij} = c_{ijkm}\varepsilon_{km}; \tag{3}$$

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$$\sigma_{ij} = c_{ijkm} s_{km}. \tag{4}$$

Meanwhile, we assume that the properties of symmetry are satisfied for the coefficients c_{ijkm} relative to permutation of the subscripts i and j and k and m, as well as the pairs ij and km. Comparison of Eqs. (3), (4), and (1) leads to the conclusion that if the canonical tensor exists, then it has certain interesting properties of duality. On the one hand, in accordance with (3), s_{ij} can be regarded as representing certain "stresses." On the other hand, in accordance with (4), it can be regarded as representing certain "strains." In order for σ_{ij} to be a real stress tensor, the constants c_{ijkm} must be linked by the relation

$$c_{ijkm} c_{kmrs} = b_{ijrs}. \tag{5}$$

Thus, finding the canonical tensor entails determination of the constants c_{ijkm}. We will assume that the coefficients c_{ijkm} can be found in the form of coefficients b_{ijkm} (2) satisfying the above-mentioned symmetry properties:

$$c_{ijkm} = \lambda_* \delta_{ij} \delta_{km} + \mu_* (\delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk}). \tag{6}$$

Here, λ_* and μ_* are unknown negative constants. Inserting (2) and (6) into (5), we have

$$[\lambda_* \delta_{ij} \delta_{km} + \mu_* (\delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk})][\lambda_* \delta_{km} \delta_{rs} + \mu_* (\delta_{kr} \delta_{ms} + \delta_{ks} \delta_{mr})] = \lambda \delta_{ij} \delta_{rs} + \mu (\delta_{ir} \delta_{js} + \delta_{is} \delta_{jr}). \tag{7}$$

With allowance for the identities $\delta_{ij}\delta_{ij} = 3$, $\delta_{ik}\delta_{km} = \delta_{im}$, $\delta_{ik} = \delta_{ki}$, we reduce Eq. (7) to the form

$$\left(3\lambda_*^2 + 4\lambda_* \mu_* - \lambda\right) \delta_{ij} \delta_{rs} + \left(2\mu_*^2 - \mu\right)(\delta_{ir} \delta_{js} + \delta_{is} \delta_{jr}) = 0,$$

from which we obtain two equations to determine λ_* and μ_*:

$$3\lambda_*^2 + 4\lambda_* \mu_* - \lambda = 0, \quad 2\mu_*^2 - \mu = 0. \tag{8}$$

In accordance with the condition adopted above, we will restrict ourselves to nonnegative solutions of Eqs. (8)

$$\lambda_* = \lambda / \left[(2\mu + 3\lambda)^{1/2} + (2\mu)^{1/2}\right], \quad \mu_* = (\mu/2)^{1/2}. \tag{9}$$

Here, $(2\mu + 3\lambda)$ is the bulk modulus of expansion. Thus, if we determine the coefficients c_{ijkm} with Eqs. (6) and (9), then the tensor s_{ij} will have properties (3) and (4). Also, it follows from (2) and (6) that all three tensors ε_{ij}, s_{ij}, and σ_{ij} are coaxial.

We will use the tensor s_{ij} to write the potential strain energy Π_* in a unit volume of the elastic body: $\Pi_* = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}c_{ijkm}s_{km}\varepsilon_{ij}$. However, in accordance with (3) and the symmetry properties of the coefficients c_{ijkm},

$$\Pi_* = \frac{1}{2} (c_{kmij} \varepsilon_{ij}) s_{km} = \frac{1}{2} s_{km}^2. \tag{10}$$

Thus, the potential energy in a unit volume of the elastic body is written in terms of the sum of the squares of the components of the canonical tensor with a multiplier of 1/2, i.e., it turns out to be reduced to canonical form [2]. The designation of the tensor as s_{ij} is in fact related to this reduction property. Due to the symmetry properties $s_{ij} = s_{ji}$, Eq. (10) contains the squares of six unknown components. It should be noted that for proof of the positive determinateness of the potential energy, it is customary [3-6] to employ a reduction to the sum of the squares of seven quantities; the possibility of a reduction to the sum of the squares of six quantities follows from the theory of quadratic forms [2]. Since the potential energy in a unit volume is a quadratic invariant, it can be written in terms of the first I_1 and second I_2 invariants of the canonical tensor: $\Pi_* = \frac{1}{2}I_1{}^2 - I_2$.

In view of the above-noted features of the tensor s_{ij}, it is natural to suppose that it has certain energy properties. Of these properties, we mention the properties of "reciprocity,"

$$\partial\Pi_*/\partial\varepsilon_{ij} = c_{ijkm} s_{km}, \quad \partial\Pi_*/\partial s_{ij} = c_{ijkm} \varepsilon_{km},$$

$$\partial\Pi_*/\partial s_{ij} = c_{ijkm}^{-1} \sigma_{km}, \quad \partial\Pi_*/\partial\sigma_{ij} = c_{ijkm}^{-1} s_{km},$$

where $c_{ijkm}{}^{-1}$ are coefficients which are the inverse of c_{ijkm} in Eqs. (3) and (4), i.e., $\varepsilon_{ij} = c_{ijkm}{}^{-1} s_{km}$, $s_{ij} = c_{ijkm}{}^{-1}\sigma_{km}$.

In the uniaxial tension of a medium (Poisson's ratio $\nu = 0$) with nonvanishing components ε_{11}, s_{11}, and σ_{11}, we have the relations $s_{11} = E^{1/2}\varepsilon_{11} = E^{-1/2}\sigma_{11}$, $\Pi_* = \frac{1}{2}s_{11}^2$ (E is the elastic modulus). As can be seen, the dimension of the components s_{ij} is equal to the square root of the stress dimension. The components s_{ij} do not have any particular physical significance, but they are measures of the intensity of the stress—strain state. The sum of the squares of these components is proportional to the energy stored in a unit volume. Thanks to the homogeneity of the energy expression in terms of s_{ij}, it is convenient to use when developing algorithms for operations connected with variation of energy in direct variational methods [7-9]. For example, canonical forms of potential energy similar to those presented above have been used in the theory of shells to obtain the stiffness matrix of flexural finite elements and to calculate geometrically nonlinear strain states of thin elastic bodies [10]. Such cases entail examination of the dependence of s_{ij} on discrete parameters q_k ($k = 1, \ldots, n$) serving as unknowns in the problem of elastic equilibrium, i.e., $s_{ij} = s_{ij}(q_1, \ldots, q_n)$. With a linear dependence of s_{ij} on q_k, the potential energy in a unit volume is written in the form

$$\Pi_* = \frac{1}{2} k_{rp} q_r q_p, \quad k_{rp} = \frac{\partial s_{ij} \partial s_{ij}}{\partial q_r \partial q_p}.$$

Here, k_{rp} are coefficients of the symmetrical stiffness matrix [8].

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